Spectral gaps in the random normal matrix model

Joakim Cronvall

Centre for Mathematical Sciences
Lund University, Sweden

Joakim.Cronvall@math.lu.se

Joint work with Yacin Ameur and Christophe Charlier

Bielefeld
December 19, 2023

Hamiltonian

The Hamiltonian of $(z_1, ..., z_n) \in \mathbb{C}^n$ is

$$H_n = \sum_{j \neq k} \log \frac{1}{|z_j - z_k|} + n \sum_{j=1}^{n} Q(z_j),$$

where

$$Q : \mathbb{C} \to \mathbb{R} \cup \{\infty\},$$

and

$$\lim_{z \to \infty} Q(z) - \log |z|^2 = +\infty.$$

The Coulomb gas

Boltzmann-Gibbs measure

$$d\mathbb{P}_n(z_1, ..., z_n) = \frac{1}{Z_n^\beta} e^{-\frac{\beta}{2} H_n} dA_n(z_1, ..., z_n)$$

where

$$Z_n^\beta = \int_{\mathbb{C}^n} e^{-\frac{\beta}{2} H_n} dA_n,$$

is the partition function and

$$dA_n(z_1, ..., z_n) = dA(z_1) \cdots dA(z_n) \quad \text{and} \quad dA = \frac{dxdy}{\pi}.$$

Weighted energy problem

As $n \to \infty$,

$$\frac{1}{n}\sum_{j=1}^{n}\delta_{z_j} \to \sigma,$$

where σ is the probability measure minimizing

$$I_Q[\mu] = \iint \log\frac{1}{|\zeta - \eta|}d\mu(\eta)d\mu(\zeta) + \int Q\,d\mu,$$

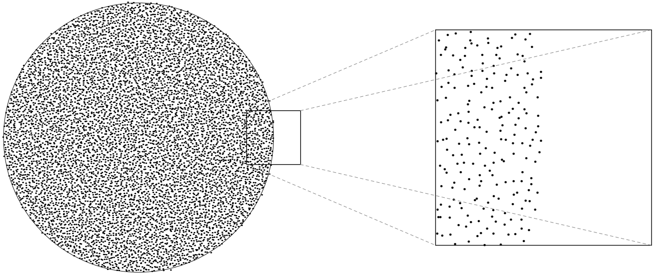
over all probability measures.

$$d\sigma(z) = 1_S \cdot \Delta Q(z)dA(z).$$

The set S is the droplet.

Ginibre

- The Ginibre case: $Q(z) = |z|^2$ ($\beta = 2$)
- $d\sigma(z) = 1_{\mathbb{D}} dA(z)$



Shallow points

- The obstacle function $\check{Q}(z)$ is the maximal subharmonic function such that $\check{Q} \leq Q$ on $\mathbb{C}$ and grows at most like $\log|z|^2 + O(1)$ at infinity.
- Coincidence set: $S^* = \{z \in \mathbb{C} : \check{Q}(z) = Q(z)\}$
- $S \subset S^*$
- Shallow points: $S^* \setminus S$

If p is a shallow point then there exists a neighborhood N such that

$$\int_{S^* \cap N} |\Delta Q| dA = 0.$$

In particular if $\Delta Q > 0$ on S^* then the shallow points have area zero.

Asymptotics of partition function

Zabrodin and Wiegman (2006) conjectured

$$\log Z_n^\beta = C_0 n^2 + C_1 n \log n + C_2 n + C_3 \log n + C_4 + O(\frac{1}{n}),$$

as $n \to \infty$.

Theorem (T. Leblé, S. Serfaty, 2018)

$$\log Z_n^\beta = n^2 C_0 + C_1 n \log n + C_2 n + o(n),$$

as $n \to \infty$.

- $C_0 = -\frac{\beta}{2} I_Q[\sigma]$
- $C_1 = \beta/4$
- $C_2 = C(\beta) - (1 - \frac{\beta}{4}) \int \log \Delta Q \, d\sigma$.

Random normal matrix model ($\beta = 2$)

Boltzmann-Gibbs measure

$$d\mathbb{P}_n(z_1, ..., z_n) = \frac{1}{Z_n} e^{-H_n} dA_n(z_1, ..., z_n)$$

where

$$Z_n = \int_{\mathbb{C}^n} e^{-H_n} dA_n.$$

Determinantal point process $\beta = 2$

For $\beta = 2$ the Coulomb gas is a determinantal point process

$$d\mathbb{P}_n(z_1, \ldots, z_n) = \frac{1}{n!} \det(K_n(z_i, z_j))_{i,j=1}^n \, dA_n(z).$$

The correlation kernel is given by

$$K_n(z, w) = \sum_{j=0}^{n-1} p_{j,n}(z) \overline{p_{j,n}(w)} e^{-\frac{n}{2}Q(z)} e^{-\frac{n}{2}Q(w)}.$$

$\{p_{j,n}\}$ is an orthonormal basis for the space of polynomials of degree at most $n - 1$ with inner product

$$(f, g) = \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-nQ(z)} \, dA(z).$$

Linear statistics

Given $h \in C_0^\infty$ we define the random variables

$$\mathrm{trace}_n(h) = \sum_{j=1}^{n} h(z_j),$$

and

$$\mathrm{fluct}_n(h) = \sum_{j=1}^{n} h(z_j) - n \int h(z) d\sigma(z).$$

Assume that

- $h \in C_0^\infty$
- Q real-analytic and $\Delta Q > 0$ in a neighborhood of the droplet
- $S^* = S$ (no shallow points)
- S consists of a single connected component

Theorem (Ameur, Hedenmalm, Makarov, 2011)

As $n \to \infty$,

$$fluct_n(h) \xrightarrow{d} N(e_h, v_h),$$

where

$$e_h = \frac{1}{2}\int_S \Delta h dA + \frac{1}{2}\int_S h\Delta L dA + \frac{1}{8\pi}\int_{\partial S} h\mathcal{N}(L^S) ds,$$

and

$$v_h = \frac{1}{4}\int |\nabla h^S|^2 dA.$$

Partition function and Fluctuations

Consider the perturbed potential

$$\tilde{Q}_{n,sh}(z) = Q(z) - \frac{1}{n} s\, h(z),$$

with Hamiltonian

$$\tilde{H}_{n,sh} = H_n - s\, \mathrm{trace}_n(h).$$

Partition functions

$$Z_n = \int e^{-H_n} dA_n, \qquad \tilde{Z}_{n,sh} = \int e^{s\, \mathrm{trace}_n h} e^{-H_n} dA_n.$$

CGF

- Consider the quotient

$$\frac{\tilde{Z}_{n,sh}}{Z_n} = \frac{1}{Z_n}\int e^{s\,\mathrm{trace}_n h} e^{-H_n} dA = \mathbb{E}_n[e^{s\,\mathrm{trace}_n(h)}]$$

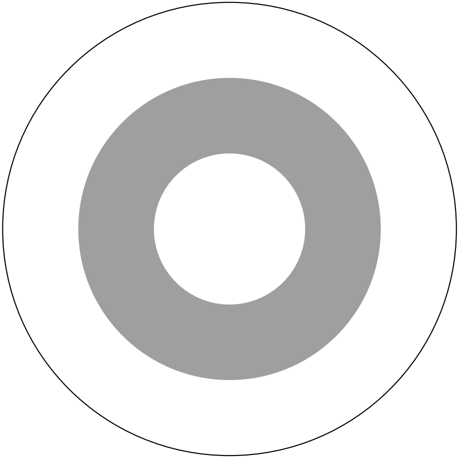
- The Cumulant generating function (CGF) of $\mathrm{trace}_n(h)$ is

$$\log \mathbb{E}_n[e^{s\,\mathrm{trace}_n(h)}] = \log \tilde{Z}_{n,sh} - \log Z_n.$$

Our setting

- Q radially symmetric potential
- $S^* = \{a \leq |z| \leq b\} \cup \{|z| = c\}$
- $\Delta Q > 0$ in a neighborhood of S^*
- h radially symmetric and $h \in \mathbb{C}_0^\infty$

Goal: Find asymptotics of the partition function and of the distribution of $\mathrm{fluct}_n(h)$ as $n \to \infty$.



Theorem (Byun, Seo, Kang, 2022)

For Q radially symmetric and $\Delta Q > 0$ in $\mathbb{C}$ the partition function can be written

$$\log Z_n = C_0 n^2 + C_1 n \log n + C_2 n + C_3 \log n + C_4 + o(1),$$

as $n \to \infty$.

- $C_0 = -I_Q[\sigma]$
- $C_1 = \frac{1}{2}$
- $C_2 = \frac{\log(2\pi)}{2} - 1 - \int \log(\Delta Q) d\sigma$
- $C_3 = \frac{6-\chi(S)}{12}$
- C_4 explicit and connected to spectral determinants

Partition function shallow points

Let $S = \{z : a \leq |z| \leq b\}$ and $S^* = S \cup \{|z| = c\}$ where $c > b$. Assume that $\Delta Q > 0$ in a neighborhood of S^*.

Theorem (Ameur, Charlier, C.)

$$\log \tilde{Z}_{n,sh} = C_0 n^2 + C_1 n \log n + \tilde{C}_2 n + C_3 \log n + \tilde{C}_4 + o(1),$$

as $n \to \infty$.

- $\tilde{C}_2 = C_2 + s \int h(z) d\sigma(z)$
- $\tilde{C}_4 = C_4 + s e_h + \frac{s^2}{2} v_h + \sum_{j=0}^{\infty} \log(1 + \mu(s)\rho^{2j+1})$.

$$\rho = \frac{b}{c} \qquad \text{and} \qquad \mu(s) = e^{s(h(c)-h(b))} \sqrt{\frac{\Delta Q(b)}{\Delta Q(c)}}.$$

Norm of orthogonal polynomials

The orthonormal polynomials are given by

$$p_j(z) = p_{j,n}(z) = \frac{z^j}{\sqrt{h_j}},$$

where

$$h_j = h_{j,n} = \int_{\mathbb{C}} |z|^{2j} e^{-nQ(z)} dA(z).$$

The partition function in a radially symmetric potential Q can be written

$$Z_n = n! \prod_{j=0}^{n-1} h_j.$$

Laplace method

$$h_j = \int\limits_{\mathbb{C}} |z|^{2j} e^{-nQ(z)} dA(z) = 2\int\limits_0^\infty r e^{-n(Q(r) - \frac{2j}{n}\log r)} dr$$

$$= 2\int\limits_0^\infty r e^{-ng_j(r)} dr,$$

where $g_j(r) = q(r) - \frac{2j}{n}\log(r)$.

Continuation of Laplace Method

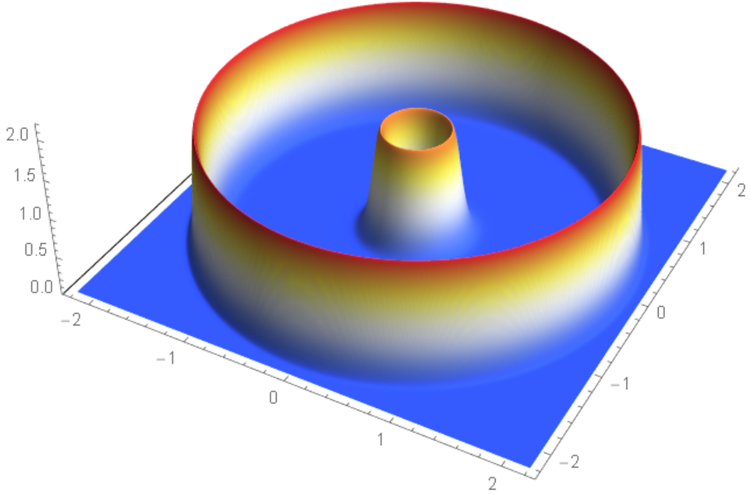
If r_* is a strict local minimum of g_j then the integral can be approximated by

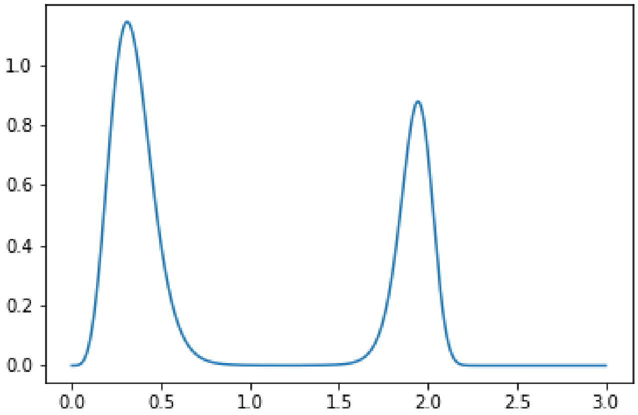
$$2\int_0^\infty re^{-ng_j(r)}dr = 2r_*\sqrt{\frac{2\pi}{n|g_j''(r_*)|}}e^{-ng_j(r_*)}(1+\frac{a_j}{n}+O(n^{-2})).$$

Polynomials with twin peaks

- The function $g_j(r)$ typically has one significant local minimum
- For j close to the critical value n there are two local minimum that become relevant
- One minimum is located near the outer boundary of the droplet and one near the shallow circle
- Let $h_{1,j}$ be the contribution from the outer boundary and $h_{2,j}$ the contribution from the shallow points

Weighted polynomial $|p_j(z)|^2 e^{-nQ(z)}$




1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0

$$\log Z_n = \log n! + \sum_{j=0}^{n-1} \log h_j.$$

$$\begin{aligned}\sum_{j=0}^{n-1} \log h_j &\approx \sum_{j=0}^{n-1} \log(h_{1,j} + h_{2,j}) \\ &= \sum_{j=0}^{n-1} \log h_{1,j} + \sum_{j=0}^{n-1} \log(1 + \frac{h_{2,j}}{h_{1,j}})\end{aligned}$$

The fraction $\frac{h_{2,j}}{h_{1,j}}$ decrease exponentially in $n - j$.

Euler-Maclaurin formula

We can approximate the sum $\sum_{j=0}^{n-1} \log h_{1,j}$ using

Theorem

If f is $2d$ times continuously differentiable on the interval $[m, n]$, then

$$\sum_{j=m}^{n-1} f(j) = \int_m^n f(x)\,dx - \frac{f(n) - f(m)}{2} + \sum_{k=1}^{d-1} \frac{B_{2k}}{(2k)!}\Big(f^{(2k-1)}(n) - f^{(2k-1)}(m)\Big) + \epsilon_d.$$

Here B_{2k} is the Bernoulli number.

CGF of Fluctuations (Shallow points)

Corollary

The cumulant generating function, $F_n(s)$, of $fluct_n(h)$ is given by

$$F_n(s) = se_h + \frac{s^2}{2}v_h + \sum_{j=0}^{\infty}\log(1+\mu(s)\rho^{2j+1}) - \sum_{j=0}^{\infty}\log(1+\mu(0)\rho^{2j+1}) + o(1),$$

as $n \to \infty$.

Binomial and Poisson distribution

- $X_{j,n}$ i.i.d. Bernoulli rv's with parameter $p_n = \frac{\lambda}{n}$.
- $Y_n \sim \sum_{j=1}^{n} X_{j,n} \sim Bin(n, p_n)$
- $Y_n \xrightarrow{d} Y \sim Poi(\lambda)$.

q-Binomial

- X_j independent Bernoulli random variables with parameter $p_j = \frac{\theta q^{j-1}}{1+\theta q^{j-1}}$, $j = 1, 2, 3...$, $0 < q < 1$ and $\theta > 0$.
- $Y_n = \sum_{j=1}^{n} X_j$
- $Y_n \sim Bin_q(n)$ has distribution

$$\mathbb{P}(X = k) = \binom{n}{k}_q \frac{q^{k(k-1)/2}\theta^k}{\prod_{j=1}^{n}(1 + \theta q^{j-1})},$$

where $\binom{n}{k}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}$.

Heine distribution

- A random variable X is Heine distributed, $X \sim He(\theta, q)$, with $0 < q < 1$ and $\theta > 0$ if its p.m.f. is given by

$$\mathbb{P}(X = k) = p_0 \frac{q^{k(k-1)/2}\theta^k}{(q;q)_k}, \qquad k = 0, 1, 2...$$

  where $(q;q)_k = \prod_{j=1}^{k}(1 - q^k)$.
- Introduced by Benkherouf and Bather in 1988.

CGF of the Heine distribution

We rewrite

$$\sum_{j=0}^{\infty} \log(1 + \mu(s)\rho^{2j+1}) = \log \sum_{k=0}^{\infty} \frac{\rho^{k^2}\mu(s)^k}{(\rho^2)_k}$$
$$= \log \sum_{k=0}^{\infty} e^{s(h(c)-h(b))k} \cdot \frac{\rho^{k^2}\mu(0)^k}{(\rho^2;\rho^2)_k}.$$

The CGF of a random variable $(h(c) - h(b)) \cdot H$ where $H \sim He(\rho \cdot \mu(0), \rho^2)$.

Fluctuations shallow point

Theorem

As $n \to \infty$,

$$fluct_n(h) \xrightarrow{d} N(e_h, v_h) + (h(c) - h(b)) \cdot H,$$

where $H \sim He(\rho \cdot \mu(0), \rho^2)$ is independent of $N(e_h, v_h)$.

Droplet with many components

- Q radially symmetric potential
- Droplet $S = \bigcup_{\nu=0}^{N} \{a_\nu \le |z| \le b_\nu\}$
- $\Delta Q > 0$ in a neighborhood of S
- $S = S^*$
- h radially symmetric and $h \in \mathbb{C}_0^\infty$

Goal: Find asymptotics of the partition function and of the distribution of $\mathrm{fluct}_n(h)$ as $n \to \infty$.

Gap ensembles

For suitable choice of

$$Q(z) = a|z|^2 - b|z|^4 + c|z|^6$$

the droplet looks as follows:

Partition function

Assume that the droplet is $S = \{a \leq |z| \leq b\} \cup \{c \leq |z| \leq d\}$.
Let $\rho = \frac{b}{c}$ and $\mu(s) = e^{s(h(c)-h(b))}\sqrt{\frac{\Delta Q(b)}{\Delta Q(c)}}$. Let $x = Mn - \lfloor Mn \rfloor$ where $M = \sigma(\{a \leq |z| \leq b\})$.

Theorem (Ameur, Charlier, C.)

As $n \to \infty$,

$$\log \tilde{Z}_{n,sh} = C_0 n^2 + C_1 n \log n + \tilde{C}_2 n + C_3 \log n + \tilde{C}_4 + o(1).$$

- $\tilde{C}_2 = C_2 + s\int h(z)d\sigma(z)$
- $\tilde{C}_4 = C_4 + se_h + \frac{s^2}{2}v_h + \sum_{j=0}^{\infty} \log(1 + \mu(s)\rho^{2(j+\frac{1}{2}+x)}) + \sum_{j=0}^{\infty} \log(1 + \mu(s)^{-1}\rho^{2(j+\frac{1}{2}-x)}) + x^2 \log \rho + x \log \mu(s).$

CGF of fluctuations (Disconnected droplet)

$$
\begin{aligned}
F_n(s) = {} & \sum_{j=0}^{\infty} \log(1 + \mu(s)\rho^{2(j+\frac{1}{2}+x)}) - \sum_{j=0}^{\infty} \log(1 + \mu(0)\rho^{2(j+\frac{1}{2}+x)}) \\
& + \sum_{j=0}^{\infty} \log(1 + \mu(s)^{-1}\rho^{2(j+\frac{1}{2}-x)}) - \sum_{j=0}^{\infty} \log(1 + \mu(0)^{-1}\rho^{2(j+\frac{1}{2}-x)}) \\
& + s e_h + \frac{s^2}{2} v_h + s x(h(c) - h(b)) + o(1),
\end{aligned}
$$

as $n \to \infty$.

Fluctuations disconnected droplet

Theorem

As $n \to \infty$,

$$\begin{aligned} fluct_n(h) \xrightarrow{d} N(e_h, v_h) &+ x(h(c) - h(b)) \\ &+ (h(c) - h(b)) \cdot H_1 + (h(b) - h(c)) \cdot H_2, \end{aligned}$$

where $H_1 \sim He(\rho^{1+2x} \cdot \mu(0), \rho^2)$, $H_2 \sim He(\rho^{1-2x} \cdot \mu(0)^{-1}, \rho^2)$ and $N(e_h, v_h)$ are independent.

Discrete normal distribution

The difference between two independent Heine distributions $He(\theta, q)$ and $He(q/\theta, q)$ is a discrete normal distribution $X \sim dN(\theta, q)$ with p.m.f.

$$\mathbb{P}(X = k) = \frac{1}{C} q^{k(k-1)/2} \theta^k$$

The discrete normal distribution minimizes entropy for specified mean and variance.

References

Main references:

- Y. Ameur, C. Charlier, J. Cronvall, The two-dimensional Coulomb gas: fluctuations through a spectral gap
- Y. Ameur, C. Charlier, J. Cronvall, Free energy and fluctuations in the random normal matrix model with spectral gaps, to appear.